GENERALIZED SUPPORT VARIETIES FOR FINITE GROUP SCHEMES

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to Andrei Suslin, with great admiration

Abstract. We construct two families of refinements of the (projectivized) support variety of a finite dimensional module M for a finite group scheme G. For an arbitrary finite group scheme, we associate a family of *non maximal rank varieties* $\Gamma^j(G)_M$, $1 \leq j \leq p-1$, to a kG-module M. For G infinitesimal, we construct a finer family of locally closed subvarieties $V^{\underline{a}}(G)_M$ of the variety of one parameter subgroups of G for any partition $\underline{a}$ of $\dim M$. For an arbitrary finite group scheme G, a kG-module M of constant rank, and a cohomology class ζ in $\mathrm{H}^1(G, M)$ we introduce the *zero locus* $Z(\zeta) \subset \Pi(G)$. We show that $Z(\zeta)$ is a closed subvariety, and relate it to the non-maximal rank varieties. We also extend the construction of $Z(\zeta)$ to an arbitrary extension class $\zeta \in \mathrm{Ext}^n_G(M, N)$ whenever M and N are kG-modules of constant Jordan type.

0. Introduction

In the remarkable papers [21], D. Quillen identified the spectrum of the (even dimensional) cohomology of a finite group $\operatorname{Spec} \mathrm{H}^\bullet(G, k)$ where k is some field of characteristic p dividing the order of the group. The variety $\operatorname{Spec} \mathrm{H}^\bullet(G, k)$ is the "control space" for certain geometric invariants of finite dimensional kG-modules. These invariants, *cohomological support varieties* and *rank varieties*, were initially introduced and studied in [1] and [6]. Over the last twenty five years, many authors have been investigating these varieties inside $\operatorname{Spec} \mathrm{H}^\bullet(G, k)$ in order to provide insights into the structure, behavior, and properties of kG-modules. The initial theory for finite groups has been extended to a much more general family of finite group schemes, starting with the work of [13] for p-restricted Lie algebras. The resulting theory of support varieties for modules for finite group schemes satisfies all of the axioms of a "support data" of tensor triangulated categories as defined in [2]. Thus, for example, this theory provides a classification of tensor–ideal, thick subcategories of the stable module category of a finite group scheme G.

In this present paper, we embark on a different perspective of geometric invariants for kG-modules for a finite group scheme G. We introduce a new family of invariants, "generalized support varieties", which stratify the support variety of a finite dimensional kG-module M. As finer invariants, they capture more structure of a module M and can distinguish between modules with the same support varieties. In particular, the generalized support varieties are always proper subvarieties

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of the control space $\operatorname{Spec} \mathrm{H}^\bullet(G,k)$ whereas the support variety often coincides with the entire control space. On the other hand, they necessarily lack certain good behavior with respect to tensor products and distinguished triangles in the stable module category of kG. However, as we shall try to convince the reader, these varieties provide interesting and useful tools in the further study of the representation theory of finite groups and their generalizations.

Since the module category of a finite group scheme G is wild except for very special G, our goals are necessarily more modest than the classification of all (finite dimensional) kG-modules. Two general themes that we follow when introducing our new varieties associated to representations are the formulation of invariants which distinguish various known classes of modules and the construction of modules with specified invariants.

In Section 1, we summarize some of our earlier work, and that of others, concerning support varieties of kG-modules. We emphasize the formulation of support varieties in terms of π-points, since the fundamental structure underlying our new invariants is the scheme $\Pi(G)$ of equivalence classes of π-points. Also in this section, we recall maximal Jordan types of kG-modules and the non-maximal subvariety $\Gamma(G)_M \subset M$ refining the support variety $\Pi(G)_M$ for a finite dimensional kG-module M.

If G is an infinitesimal group scheme, one formulation of support varieties is in terms of the affine scheme $V(G)$ of infinitesimal subgroups of G. For any Jordan type $\underline{a} = \sum_{i=1}^p a_i[i]$ and any finite dimensional kG-module M (with G infinitesimal), we associate in Section 2 subvarieties $V^{\leq \underline{a}}(G)_M$ and $V^{\underline{a}}(G)_M$ of $V(G)$. Determination of these refined support varieties is enabled by earlier computations of the global p-nilpotent operator $\Theta_G : M \otimes k[V(G)] \to M \otimes k[V(G)]$ which was introduced and studied in [17].

We require a refinement of one of the main theorems of [18] recalled as Theorem 1.5. Section 3 outlines the original proof due to A. Suslin and the authors, and points out the minor modifications required to establish the fact that whether or not a kG-module has maximal j-type at a π-point depends only upon the equivalence class of that π-point (Theorem 3.6). This is the key result needed to establish that the generalized support varieties are well–defined for all finite group schemes.

In Section 4, we consider closed subvarieties $\Gamma^j(G)_M \subset \Pi(G)$ for any finite group scheme, finite dimensional kG-module M, and integer $j, 1 \leq j < p$, *the non maximal rank varieties*. We establish some properties of these varieties and work out a few examples to suggest how these invariants can distinguish certain non-isomorphic kG-modules.

In the concluding Section 5, we employ π-points to associate a closed subvariety $Z(\zeta) \subset \Pi(G)$ to a cohomology class $\zeta \in \mathrm{H}^1(G,M)$ provided that M is a kG-module of constant rank. One of the key properties of $Z(\zeta)$ is that $Z(\zeta) = \emptyset$ if and only if the extension $0 \to M \to E_\zeta \to k \to 0$ satisfies the condition that E_ζ is also a kG-module of constant rank. We show that $Z(\zeta)$ is often homeomorphic to $\Gamma^1(G)_{E_\zeta}$ which allows us to conclude that $Z(\zeta)$ is closed. Taking M to be an odd degree Heller shift of the trivial module k, we recover the familiar zero locus of a class in $\mathrm{H}^{2n}(G,k)$ in the special case $M = k$. Finally, we generalize this construction to extension classes $\xi \in \operatorname{Ext}^n_G(M,N)$ for kG-modules M and N of constant Jordan type and any $n \geq 0$.

We abuse terminology in this paper by referring to a (Zariski) closed subset of an affine or projective variety as a subvariety. Should one wish, one could always impose the reduced scheme structure on such "subvarieties".

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1. Recollection of Π-point schemes and support varieties

Throughout, k will denote an arbitrary field of characteristic $p > 0$. Unless explicit mention is made to the contrary, G will denote a finite group scheme over k with finite dimensional coordinate algebra $k[G]$. We denote by kG the Hopf algebra dual to $k[G]$, and refer to kG as the group algebra of G. Thus, (left) kG-modules are naturally equivalent to (left) $k[G]$-comodules, which are equivalent to (left) rational G-modules (see [20, ch.1]). If M is a kG-module and K/k is a field extension, then we denote by M_K the KG-module obtained by base change.

We shall identify $\mathrm{H}^*(G,k)$ with $\mathrm{H}^*(kG,k)$.

Definition 1.1. ([16]) The Π-*point scheme* of a finite group scheme G is the k-scheme of finite type whose points are equivalence classes of π-points of G and whose scheme structure is defined in terms of the category of kG-modules.

In more detail,

(1) A π-point of G is a (left) flat map of K-algebras $\alpha_K : K[t]/t^p \to KG$ for some field extension K/k with the property that there exists a unipotent abelian subgroup scheme $i : C_K \subset G_K$ defined over K such that α_K factors through $i_* : KC_K \to KG_K = KG$.

(2) If $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$ are two π-points of G, then α_K is said to be a *specialization* of β_L, provided that for any finite dimensional kG-module M, $\alpha_K^*(M_K)$ being free as $K[t]/t^p$-module implies that $\beta_L^*(M_L)$ is free as $L[t]/t^p$-module.

(3) Two π-points $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$ are said to be *equivalent*, written $\alpha_K \sim \beta_L$, if they satisfy the following condition for all finite dimensional kG-modules M: $\alpha_K^*(M_K)$ is free as $K[t]/t^p$-module if and only if $\beta_L^*(M_L)$ is free as $L[t]/t^p$-module.

(4) A subset $V \subset \Pi(G)$ is closed if and only if there exists a finite dimensional kG-module M such that V equals

$$\Pi(G)_M = \{[\alpha_K] \,|\, \alpha_K^*(M_K) \text{ is not free as } K[t]/t^p - \text{module}\}$$

The closed subset $\Pi(G)_M \subset \Pi(G)$ is called the Π-*support of* M.

(5) The topological space $\Pi(G)$ of equivalence classes of π-points can be endowed with a scheme structure based on representation theoretic properties of G (see [16, §7]).

We denote by

$$\mathrm{H}^\bullet(G,k) = \begin{cases} \mathrm{H}^*(G,k), & \text{if } p = 2, \\ \mathrm{H}^{\mathrm{ev}}(G,k) & \text{if } p > 2. \end{cases}$$

The *cohomological support variety* $|G|_M$ of a kG-module M is the closed subspace of $\operatorname{Spec} \mathrm{H}^\bullet(G,k)$ defined as the variety of the ideal $\operatorname{Ann}_{\mathrm{H}^\bullet(G,k)} \operatorname{Ext}^*_G(M,M) \subset \mathrm{H}^\bullet(G,k)$.

Theorem 1.2. [16, 7.5] *Let G be a finite group scheme, and M be a finite dimensional kG-module. Denote by $\operatorname{Proj} \mathrm{H}^\bullet(G,k)$ the projective k-scheme associated to the commutative, graded k-algebra $\mathrm{H}^\bullet(G,k)$. Then there is an isomorphism of k-schemes*

$$\Phi_G : \operatorname{Proj} \mathrm{H}^\bullet(G,k) \simeq \Pi(G)$$

which restricts to a homeomorphism of closed subspaces

$$\operatorname{Proj}(|G|_M) \simeq \Pi(G)_M$$

for all finite dimensional kG-modules M.

We (implicitly) identify $\operatorname{Proj} \mathrm{H}^\bullet(G,k)$ with $\Pi(G)$ via this isomorphism.

We consider the *stable module category* $\operatorname{stmod} kG$. Recall that the Heller shift $\Omega(M)$ of M is the kernel of the minimal projective cover $P(M) \twoheadrightarrow M$, and the inverse Heller shift $\Omega^{-1}(M)$ is the cokernel of the embedding of M into its injective hull, $M \hookrightarrow I(M)$.

The objects of $\operatorname{stmod} kG$ are finite dimensional kG-modules. The morphisms are equivalence classes where two morphisms are equivalent if they differ by a morphism which factors through a projective module,

$$\operatorname{Hom}_{\operatorname{stmod} kG}(M,N) = \operatorname{Hom}_{kG}(M,N)/\operatorname{PHom}_{kG}(M,N).$$

The stable module category has a tensor triangulated structure: the triangles are induced by exact sequences, the shift operator is given by the inverse Heller operator Ω^{-1}, and the tensor product is the standard tensor product in the category of kG-modules. Two kG-modules M, N are stably isomorphic if and only if they are isomorphic as kG-modules up to a projective direct summand.

The association $M \mapsto \Pi(G)_M$ fits the abstractly defined "theory of supports" for the stable module category of G (as defined in [2]). Some of the basic properties of this theory are summarized in the next theorem (see [16]).

Theorem 1.3. *Let G be a finite group scheme and let M, N be finite dimensional kG-modules.*

(1) $\Pi(G)_M = \emptyset$ *if and only if M is projective as a kG-module.*
(2) $\Pi(G)_{M\oplus N} = \Pi(G)_M \cup \Pi(G)_N$.
(3) $\Pi(G)_{M\otimes N} = \Pi(G)_M \cap \Pi(G)_N$.
(4) $\Pi(G)_M = \Pi(G)_{\Omega M}$.
(5) *If $M \to N \to Q \to \Omega^{-1}M$ is an exact triangle in the stable module category* $\operatorname{stmod}(kG)$ *then* $\Pi(G)_N \subset \Pi(G)_M \cup \Pi(G)_Q$.
(6) *If p does not divide the dimension of M, then* $\Pi(G)_M = \Pi(G)$.

The last property of Theorem 1.3 indicates that $M \mapsto \Pi(G)_M$ is a somewhat crude invariant.

We next recall the use of Jordan types in order to refine this theory. The isomorphism type of a finite dimensional $k[t]/t^p$-module M is said to be the Jordan type of M. We denote the Jordan type of M by $\operatorname{JType}(M)$, and write $\operatorname{JType}(M) = \sum_{i=1}^p a_i[i]$; in other words, as a $k[t]/t^p$-module $M \simeq \bigoplus_{i=1}^p ([i])^{\oplus a_i}$ where $[i] = k[t]/t^i$. Thus, we may (and will) view a Jordan type $\operatorname{JType}(M)$ as a partition of $m = \dim M$ into subsets each of which has cardinality $\leq p$.

We shall compare Jordan types using the *dominance order*. Let $\underline{n} = [n_1 \leq n_2 \leq \ldots \leq n_k]$, $\underline{m} = [m_1 \leq m_2 \leq \ldots \leq m_k]$ be two partitions of N. Then $\underline{n}$ *dominates* $\underline{m}$, written $\underline{n} \geq \underline{m}$, iff

$$\sum_{i=j}^{k} n_i \ \geq \ \sum_{i=j}^{k} m_i. \tag{1.3.1}$$

for all j, $1 \leq j \leq k$. For $k[t]/t^p$-modules M, N, we say that $\mathrm{JType}(M) \geq \mathrm{JType}(N)$ if the partition corresponding to $\mathrm{JType}(M)$ dominates the partition corresponding to $\mathrm{JType}(N)$. The dominance order on Jordan types can be reformulated in the following way.

Lemma 1.4. *Let M, N be $k[t]/t^p$-modules of dimension m. Then* $\mathrm{JType}(M) \geq \mathrm{JType}(N)$ *if and only if*

$$\mathrm{rk}(t^j, M) \geq \mathrm{rk}(t^j, N)$$

for all j, $1 \leq j < p$, where $\mathrm{rk}(t^j, M)$ denotes the rank of the operator t^j on M.

Proof. If $\mathrm{JType}(M) = \sum\limits_{i=1}^{p} a_i[i]$, then

$$\mathrm{rk}(t^j, M) = \sum_{i=j+1}^{p} a_i(i-j). \tag{1.4.1}$$

The statement now follows from [10, 6.2.2]. □

The following theorem plays a key role in our formulation of geometric invariants for a kG-module M that are finer than the Π-support $\Pi(G)_M$. In Section 3, we outline the proof of this theorem in order to prove the related, but sharper, Theorem 3.6. We say that a π-point α_K has maximal Jordan type on a kG-module M if there does not exist a π-point β_L such that $\mathrm{JType}(\alpha_K^*(M_K)) < \mathrm{JType}(\beta_L^*(M_L))$.

Theorem 1.5. [18, 4.10] *Let G be a finite group scheme over k and M a finite dimensional kG-module. Let $\alpha_K : K[t]/t^p \to KG$ be a π-point of G which has maximal Jordan type on M. Then for any π-point $\beta_L : L[t]/t^p \to LG$ which specializes to α_K, the Jordan type of $\alpha_K^*(M_K)$ equals the Jordan type of $\beta_L^*(M_L)$; in particular, if $\alpha_K \sim \beta_L$, then the Jordan type of $\alpha_K^*(M_K)$ equals the Jordan type of $\beta_L^*(M_L)$.*

The following class of kG-modules was introduced in [8] and further studied in [7], [9], [4], [5].

Definition 1.6. A finite dimensional kG-module M is said to be of *constant Jordan type* if the Jordan type of $\alpha_K^*(M_K)$ is the same for every π-point α_K of G. By Theorem 1.5, M has constant Jordan type $\underline{a}$ if and only if for each point of $\Pi(G)$ there is some representative α_K of that point with $\mathrm{JType}(M) = \underline{a}$.

Theorem 1.5 justifies the following definition (see [18, 5.1]).

Definition 1.7. ([18, 5.1]) Let M be a finite dimensional representation of a finite group scheme G. We define $\Gamma(G)_M \subset \Pi(G)$ to be the subset of equivalence classes of π-points $\alpha_K : K[t]/t^p \to KG$ such that $\mathrm{JType}(\alpha_K^*(M_K))$ is not maximal among Jordan types $\mathrm{JType}(\beta_L^*(M_L))$ where β_L runs over all π-points of G.

To conclude this summary, we recall certain properties of the association $M \mapsto \Gamma(G)_M$.

Proposition 1.8. *Let G be a finite group scheme and let M, N be finite dimensional kG-modules. Then $\Gamma(G)_M \subset \Pi(G)$ is a closed subvariety satisfying the following properties:*

1. *If M and N are stably isomorphic, then $\Gamma(G)_M = \Gamma(G)_N$.*
2. *$\Gamma(G)_M \subset \Pi(G)_M$ with equality if and only if $\Pi(G)_M \neq \Pi(G)$.*
3. *$\Gamma(G)_M$ is empty if and only if M has constant Jordan type.*
4. *If M has constant Jordan type, then $\Gamma(G)_{M\oplus N} = \Gamma(G)_N$.*
5. *If $\Pi(G)$ is irreducible, then N has constant non-projective Jordan type if and only if $\Gamma(G)_{M\otimes N} = \Gamma(G)_M$ for any kG-module M.*
6. *If $\Pi(G)$ is irreducible, then*
$$\Gamma(G)_{M\otimes N} \;=\; (\Gamma(G)_M \cup \Gamma(G)_N) \cap (\Pi(G)_M \cap \Pi(G)_N).$$

Proof. If M and N are stably isomorphic then $M = N \oplus P$ (or vice versa) with P projective. Since projective modules have constant Jordan type, (1) becomes a special case of (4). The fact that $\Gamma(G)_M \subset \Pi(G)$ is closed is proved in [18, 5.2]. Properties (2) and (3) follow essentially from definitions. Property (4) follows from the additivity of the dominance order. Properties (5) and (6) are the statements of [8, 4.9] and [8, 4.7] respectively. □

2. Refined support varieties for infinitesimal group schemes

Before considering refinements of $\Gamma(G)_M \subset \Pi(G)$ in Section 3 for a general finite group scheme G, we specialize in this section to infinitesimal group schemes and work with the affine variety $V(G)$. First, we recall some definitions and several fundamental results from [23], [24].

A finite group scheme is called *infinitesimal* if its coordinate algebra $k[G]$ is local. Important examples of infinitesimal group schemes are Frobenius kernels of algebraic groups (see [20]). An infinitesimal group scheme is said to have height less or equal to r if for any x in $\mathrm{Rad}(k[G])$, $x^{p^r} = 0$. A *one-parameter subgroup* of height r of G over a commutative k-algebra A is a map of group schemes over A of the form $\mu : \mathbb{G}_{a(r),A} \to G_A$. Here, $\mathbb{G}_{a(r),A}$, G_A are group schemes over A defined as the base changes from k to A of $\mathbb{G}_{a(r)}$, G.

Let $\mathbb{G}_a$ be the additive group, and $\mathbb{G}_{a(r)}$ be the r-th Frobenius kernel of $\mathbb{G}_a$. Then $k[\mathbb{G}_{a(r)}] = k[T]/T^{p^r}$, and $k\mathbb{G}_{a(r)} = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$, indexed so that the Frobenius map $F : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ satisfies $F_*(u_i) = u_{i-1}, i > 0; F_*(u_0) = 0$. We define

$$\epsilon : k[u]/u^p \;\to\; k\mathbb{G}_{a(r)} = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p) \tag{2.0.1}$$

to be the map sending u to $u_{r-1} \in k\mathbb{G}_{a(r)}$. Thus, ϵ is a map of group algebras but not of Hopf algebras in general. In fact, the map ϵ is induced by a group scheme homomorphism if and only if $r = 1$ in which case ϵ is an isomorphism.

Theorem 2.1. [23] *Let G be an infinitesimal group scheme of height $\leq r$. Then there is an affine group scheme $V(G)$ which represents the functor sending a commutative k-algebra A to the set $\mathrm{Hom}_{\mathrm{gr.sch}/A}(\mathbb{G}_{a(r),A}, G_A)$.*

Thus, a point $v \in V(G)$ naturally corresponds to a 1-parameter subgroup

$$\mu_v : \mathbb{G}_{a(r),k(v)} \longrightarrow G_{k(v)}$$

where $k(v)$ is the residue field of v.

Theorem 2.2. [24] *(1). The closed subspaces of $V(G)$ are the subsets of the form*

$$V(G)_M = \{v \in V(G) \,|\, \epsilon^*\mu_v^*(M_{k(s)}) \text{ is not free as a module over } k(v)[u]/u^p\}$$

for some finite dimensional kG-module M.
(2). There is a natural p-isogeny $V(G) \longrightarrow \operatorname{Spec} \mathrm{H}^\bullet(G,k)$ which restricts to a homeomorphism $V(G)_M \simeq |G|_M$ for any finite dimensional kG-module M.

Theorem 1.2 implies that the spaces $\Pi(G)$ and $\operatorname{Proj} k[V(G)]$ are also homeomorphic (see [16] for a natural direct relationship between $\Pi(G)$ and $V(G)$ for an infinitesimal group scheme).

Let $\mu_{v*} : k(v)\mathbb{G}_{a(r)} \to k(v)G$ be the map on group algebras induced by the one-parameter subgroup $\mu_v : \mathbb{G}_{a(r)} \to G$. We denote by θ_v the nilpotent element of $k(v)G$ which is the image u under the composition

$$k(v)[u]/u^p \xrightarrow{\epsilon} k(v)[u_0,\dots,u_{r-1}]/(u_0^p,\dots,u_{r-1}^p) \xrightarrow{\mu_{v*}} k(v)G\ .$$

So, $\theta_v = \mu_{v*}(u_{r-1}) \in k(v)G$. For a given kG-module M we also let

$$\theta_v : M_{k(v)} \to M_{k(v)}$$

denote the associated p-nilpotent endomorphism. Thus, $\mathrm{JType}(\epsilon^*\mu_v^*(M_{k(v)}))$ is the Jordan type of θ_v on $M_{k(v)}$.

Definition 2.3. Let M be a kG-module of dimension m. We define the *local Jordan type function*

$$\mathrm{JType}_M : V(G) \ \to \ \mathbb{N}^{\times p}, \tag{2.3.1}$$

by sending v to $(a_1,\dots,a_p)$, where $(\theta_v)^*(M_{k(v)}) \simeq \sum_{i=1}^p a_i[i]$.

Definition 2.4. For a given $\underline{a} = (a_1,\dots,a_p) \in \mathbb{N}^{\times p}$, we define

$$V^{\underline{a}}(G)_M = \{v \in V(G) \,|\, \mathrm{JType}_M(v) = \underline{a}\},$$

$$V^{\leq \underline{a}}(G)_M = \{v \in V(G) \,|\, \mathrm{JType}_M(v) \leq \underline{a}\}.$$

As we see in the following example, $V^{\underline{a}}(G)_M$ is a generalization of a nilpotent orbit of the adjoint representation (and $V^{\leq\underline{a}}(G)_M$ is a generalization of an orbit closure).

Example 2.5. Let $G = \mathrm{GL}_{N(1)}$ and let M be the standard N-dimensional representation of GL_N. Then JType_M sends a p-nilpotent matrix X to its Jordan Jordan type as an endomorphism of M. Consequently, JType_M has image inside $\mathbb{N}^{\times p}$ consisting of those p-tuples $\underline{a} = (a_1,\dots,a_p)$ such that $\sum_i a_i \cdot i = N$. The locally closed subvarieties $V^{\underline{a}}(G)_M \subset \mathcal{N}_p(\mathfrak{gl}_N)$ are precisely the adjoint GL_N-orbits inside the p-nilpotent cone $\mathcal{N}_p(\mathfrak{gl}_N)$ of the Lie algebra $\mathfrak{gl}_N$.

Example 2.6. Let $\zeta \in \mathrm{H}^{2i+1}(G,k)$ be a non-zero cohomology class of odd degree. Let L_ζ be the Carlson module defined as the kernel of the map $\Omega^{2i+1}(k) \to k$ corresponding to ζ (see [3, II.5.9]). The module $\Omega^{2i+1}(k)$ has constant Jordan type $m[p] + [p-1]$. Let $\underline{a} = m[p] + [p-2]$ and $\underline{b} = (m-1)[p] + 2[p-1]$. Then the image of JType_{L_ζ} equals $\{\underline{a}, \underline{b}\} \subset \mathbb{N}^{\times p}$. Moreover, $V^{\underline{a}}(G)_{L_\zeta}$ is open in $V(G)$, with complement $V^{\underline{b}}(G)_{L_\zeta}$.

Remark 2.7. An explicit determination of the global p-nilpotent operator $\Theta_M : M \otimes k[V(G)] \to M \otimes k[V(G)]$ of [17, 2.4] immediately determines the local Jordan type function JType_M. Namely, to any $v \in V(G)$ we associate a nilpotent linear operator $\theta_v : M_{k(v)} \to M_{k(v)}$ defined by $\theta_v = \Theta_M \otimes_{k(v)[V(G)]} k(v)$. The local Jordan type of M at the point v is precisely the Jordan type of the linear operator θ_v.

The reader should consult [17] for many explicit examples of kG-modules M for each of the four families of examples of infinitesimal group schemes: (i.) G of height 1, so that M is a p-restricted module for $Lie(G)$; (ii.) $G = \mathbb{G}_{a(r)}$; (iii.) $\mathrm{GL}_{n(r)}$; and (iv.) $\mathrm{SL}_{2(2)}$.

We provide a few elementary properties of these refined support varieties.

Proposition 2.8. *Let M be a kG-module of dimension m and let $\underline{a} = (a_1, \ldots, a_p)$ such that $\sum_{i=1}^p a_i \cdot i = m$.*

1. *If $m = p \cdot m'$, then $V(G) \setminus V(G)_M = V^{(0,\ldots,0,m')}(G)_M$; otherwise, $V(G) = V(G)_M$.*
2. *M has constant Jordan type if and only if $V(G)_M = V^{\underline{a}}(G)_M$ for some $\underline{a} \in \mathbb{N}^{\times p}$ (in which case $\underline{a}$ is the Jordan type of M).*
3. *$V^{\leq \underline{a}}(G)_M = \{v \in V(G) \mid \mathrm{JType}_M(v) \leq \underline{a}\}$ is a closed subvariety of $V(G)$.*
4. *$V^{\underline{a}}(G)_M$ is a locally closed subvariety of $V(G)$, open in $V^{\leq \underline{a}}(G)_M$.*
5. *$V^{\leq \underline{b}}(G)_M \subseteq V^{\leq \underline{a}}(G)_M$, if $\underline{b} \leq \underline{a}$, where " $\leq$ " is the dominance order on Jordan types.*

Proof. Properties (1) and (2) follow immediately from the definitions of $V(G)_M$ in Theorem 2.1 and of constant Jordan type in Definition 1.6. Property (5) is immediate.

To prove (3), we utilize $\theta_v = \Theta_M \otimes_{k(v)[V(G)]} k(v) : M_{k(v)} \to M_{k(v)}$ described in Remark 2.7. Applying Nakayama's Lemma as in [17, 4.11] to $\mathrm{Ker}\{\Theta_M^j\}$, $1 \leq j < p$, we conclude that $\mathrm{rk}(\theta_v^j, M), 1 \leq j \leq p-1$, is lower semi-continuous. Consequently, (1.3.1) and Lemma 1.4 imply that $V^{\leq \underline{a}}(G)_M$ is closed.

Property (4) follows from the observation that $V^{\underline{a}}(G)_M$ is the complement inside $V^{\leq \underline{a}}(G)_M$ of the finite union $V^{< \underline{a}}(G)_M = \cup_{\underline{a}' < \underline{a}} V^{\leq \underline{a}'}$, which is closed by (3). □

It is often convenient to consider the *stable Jordan type* of a $k[t]/t^p$-module M: if $a_1[1] + \ldots + a_p[p]$ is the Jordan type of M, then the stable Jordan type of M is $a_1[1] + \ldots + a_{p-1}[p-1]$ (equivalently, the isomorphism class of M in the stable module category $\mathrm{stmod}\, k[u]/u^p$). We define the stable local Jordan type function

$$\underline{\mathrm{JType}}_M : V(G) \to \mathbb{N}^{\times p-1}, \quad v \mapsto (a_1, \ldots, a_{p-1})$$

by sending v to the stable Jordan type of $\theta_v^*(M_{k(v)})$.

The following proposition relates the Jordan type function for a module M and its Heller twist.

Proposition 2.9. *For a stable Jordan type $\underline{a} = \sum_{i=1}^{p-1} a_i[i]$, denote by $\underline{a}^\perp$ the "flip" of $\underline{a}$,*

$$\underline{a}^\perp = \sum_{i=1}^{p-1} a_{p-i}[i].$$

Then

$$\underline{\mathrm{JType}}_{\Omega(M)}(v) = \underline{\mathrm{JType}}_M(v)^\perp, \quad v \in V(G).$$

Proof. For any $v \in V(G)$, $\mu_v^* : (k(v)G - \text{mod}) \to (k(v)\mathbb{G}_{a(r)} - \text{mod})$ is exact. Moreover, $\epsilon^* : (k\mathbb{G}_{a(r)} - \text{mod}) \to (k[u]/u^p - \text{mod})$ is also exact. Consequently, the existence of a short exact sequence of the form $0 \to \Omega M \to P \to M \to 0$ with $\text{JType}_P(v) = N[p]$ for some N implies the assertion. □

Example 2.10. Let g be a restricted Lie algebra with restricted enveloping algebra $u(g)$ (which is isomorphic to the group algebra of an infinitesimal group scheme of height 1). Let ζ be an even dimensional cohomology class in $\mathrm{H}^\bullet(u(g), k)$, and L_ζ be the Carlson module defined by ζ. Then L_ζ has two local Jordan types: it is generically projective (that is, the local Jordan type is $m[p]$ on a dense open set), and has the type $r[p] + [p-1] + [1]$ on the hypersurface $\langle \zeta = 0 \rangle$ in $\operatorname{Spec} \mathrm{H}^\bullet(u(g), k)$. Let M be a g-module of constant Jordan type $\underline{a}$. Then the module $L_\zeta \otimes M$ has two local Jordan types: it is generically projective, and has the "stably palindromic" type $\underline{a} + \underline{a}^\perp + [\text{proj}]$ on $\langle \zeta = 0 \rangle$.

We conclude this section with the following cautionary example which shows why the construction of our local Jordan type function does not apply to kG-modules M for finite groups G.

Example 2.11. ([18, 2.3]) Let $E = \mathbb{Z}/p \times \mathbb{Z}/p$, and write $kE = k[x, y]/(x^p, y^p)$. Let $M = kE/(x - y^2)$. Then

$$\alpha : k[t]/t^p \to kE, \quad t \mapsto x$$

and

$$\alpha' : k[t]/t^p \to kE, \quad t \mapsto x - y^2$$

are equivalent as π-points of E. However, the Jordan type of $\alpha^*(M)$ equals $[\frac{p-1}{2}] + [\frac{p+1}{2}]$, whereas the Jordan type of $\alpha'^*(M)$ is $p[1]$.

3. Maximal j–rank for arbitrary finite group schemes

We begin with the following definition.

Definition 3.1. Let G be a finite group scheme, $\alpha_K : K[t]/t^p \to KG$ be a π-point of G, and j a positive integer with $1 \le j < p$. Then α_K is said to be of maximal j-rank for some finite-dimensional kG-module M provided that the rank of $\alpha_K(t^j) = \alpha_K(t)^j : M_K \to M_K$ is greater or equal to the rank of $\beta_L(t^j) : M_L \to M_L$ for any π-point $\beta_L : L[t]/t^p \to LG$.

The purpose of this section is to establish in Theorem 3.6 that maximality of j-rank at α_K implies maximal j-rank at β_L for any $\beta_L \sim \alpha_K$. The proof consists of repeating almost verbatim the proof by A. Suslin and the authors in [18] of Theorem 1.5, so that we merely indicate here the explicit places at which the proof of Theorem 1.5 should be modified in order to prove Theorem 3.6.

The following theorem provides the key step.

Theorem 3.2. *Let k be an infinite field, M be a finite-dimensional k-vector space, and $\alpha, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be a family of commuting nilpotent k-linear endomorphisms of M. Let $1 \le j \le p-1$, and assume that*

$$\operatorname{rk} \alpha^j \ge \operatorname{rk}(\alpha + \lambda_1\alpha_1 + \ldots + \lambda_n\alpha_n)^j$$

for any field extension K/k and any n-tuple $(\lambda_1, \ldots, \lambda_n) \in K^n$. Then

$$\operatorname{rk} \alpha^j = \operatorname{rk}(\alpha + \alpha_1\beta_1 + \ldots + \alpha_n\beta_n)^j.$$

In particular, if $p(x, x_1, \ldots, x_n)$ is any polynomial without constant or linear term then

$$\operatorname{rk} \alpha^j = \operatorname{rk}(\alpha + p(\alpha, \alpha_1, \ldots, \alpha_n))^j.$$

Proof. For $j = 1$, this is [18, 1.8]. For general j, the statement follows by applying Corollary 1.11 of [18]. $\square$

For any π-point $\alpha_K : K[t]/t^p \to KG$, we denote by $\operatorname{rk}(\alpha_K(t^j), M_K)$ the rank of the K-linear endomorphism $\alpha_K(t^j) : M_K \to M_K$.

In the next 3 propositions, we consider the special cases in which G is an elementary abelian p-group, an abelian finite group scheme, and an infinitesimal finite group scheme. In this manner, we follow the strategy of the proof of Theorem 1.5.

Proposition 3.3. *Let E be an elementary abelian p-group of rank r, let M be a finite dimensional kE-module, and let α_K be a π-point of E which is of maximal j-rank for M. Then for any $\beta_L \sim \alpha_K$,*

$$\operatorname{rk}(\alpha_K(t^j), M_K) = \operatorname{rk}(\beta_L(t^j), M_L).$$

Proof. The proof of [18, 2.7] applies verbatim provided one replaces references to [18, 1.12] by references to [18, 1.9]. $\square$

Proposition 3.4. *Let C be an abelian finite group scheme over k, let M be a finite dimensional kC-module, and let α_K be a π-point of C which is of maximal j-rank for M. Then for any $\beta_L \sim \alpha_K$,*

$$\operatorname{rk}(\alpha_K(t^j), M_K) = \operatorname{rk}(\beta_L(t^j), M_L).$$

Proof. The proof of [18, 2.9] applies verbatim provided one replaces references to [18, 2.7] by references to Proposition 3.3 and references to [18, 1.12] by references to Theorem 3.2. $\square$

Proposition 3.5. *Let G be an infinitesimal group scheme over k and let M be a finite dimensional kG-module. Let $\beta_L : L[t]/t^p \to LG$ be a π-point of G with the property that the j-rank of $\beta_L^*(M_L)$ is maximal for M. Then for any π-point $\alpha_K : K[t]/t^p \to KG$ which specializes to β_L,*

$$\operatorname{rk}(\alpha_K(t^j), M_K) = \operatorname{rk}(\beta_L(t^j), M_L).$$

Proof. The proof of [18, 3.5] applies verbatim provided one replaces references to [18, 2.9] by references to Proposition 3.4. $\square$

We now state and prove the assertion that maximality of j-rank at α_K implies maximality of j-rank at β_L for any $\beta_L \sim \alpha_K$. This statement for all $j, 1 \leq j < p$, implies the maximality of Jordan type as asserted in Theorem 1.5.

Theorem 3.6. *Let G be a finite group scheme over k and let M be a finite dimensional kG-module. Let $\alpha_K : K[t]/t^p \to KG$ be a π-point of G which is of maximal j-rank for M. Then for any π-point $\beta_L : L[t]/t^p \to LG$ that specializes to α_K, we have*

$$\operatorname{rk}(\alpha_K(t^j), M_K) = \operatorname{rk}(\beta_L(t^j), M_L).$$

Proof. The proof of [18, 4.10] applies verbatim provided one replaces references to [18, 2.9] by references to Proposition 3.4 and references to [18, 3.5] by references to Proposition 3.5. $\square$

We can now generalize the *modules of constant j-rank* as defined for infinitesimal group schemes in [17] to all finite group schemes.

Definition 3.7. A finite dimensional kG-module M is said to be of *constant j-rank*, $1 \leq j < p$, if for any two π-points $\alpha_K : K[t]/t^p \to KG$, $\beta_L : L[t]/t^p \to LG$, we have

$$\operatorname{rk}(\alpha_K(t^j), M_K) = \operatorname{rk}(\beta_L(t^j), M_L).$$

Remark 3.8. By Theorem 3.6, M has constant j-rank n if and only if for each point of $\Pi(G)$ there is some π-point representative α_K with $\operatorname{rk}(\alpha_K(t^j), M_K) = n$.

Evidently, a kG-module has constant Jordan type if and only if it has constant j-rank for all $j, 1 \leq j < p$ (see (1.3.1)).

We shall say that M is a *module of constant rank* if it has constant 1-rank. Every module of constant Jordan type has, by definition, constant rank. On the other hand, there are numerous examples of modules of constant rank which do not have constant Jordan type. For example, if $\zeta \in \mathrm{H}^{2i+1}(G, k)$ is non-zero and $p > 2$, then the Carlson module L_ζ is a kG-module of constant rank but not of constant Jordan type.

We finish this section with a cautionary example that illustrates that not all properties of maximal or constant Jordan type have natural analogues for maximal or constant rank. Recall that a *generic Jordan type* of a kG-module M is the Jordan type at a π-point which represents a generic point of $\Pi(G)$. By the main theorem of [18], it is well-defined. If $\Pi(G)$ is irreducible, we can therefore refer to the generic Jordan type of M. We can similarly define a *generic j-rank* of a kG-module to be $\operatorname{rk}(\alpha_K(t^j), M_K)$ for a π-point α of G representing a generic point of $\Pi(G)$. By [18, 4.2], generic j-rank is well-defined.

Example 3.9. Throughout this example we are using the formula for the tensor product of Jordan types (see, for example, [8, Appendix]).

(1). Let $\underline{a} = \sum a_i[i], \underline{b} = \sum b_i[i]$ be two Jordan types (or partitions) of the same cardinality. In [8, 4.1] the authors showed that $\underline{a} \geq \underline{b}$ implies $\underline{a} \otimes \underline{c} \geq \underline{b} \otimes \underline{c}$ for any Jordan type $\underline{c}$. The analogous statement is not true for ranks.

Indeed, let $\underline{a} = 3[2]$, $\underline{b} = [3] + 3[1]$, and $\underline{c} = [2]$. Then

$$\operatorname{rk} \underline{a} = 3 > \operatorname{rk} \underline{b} = 2.$$

Since $\underline{a} \otimes \underline{c} = 3[3] + 3[1]$ and $\underline{b} \otimes \underline{c} = [4] + 4[2]$, we have

$$\operatorname{rk} \underline{a} \otimes \underline{c} = 6 < \operatorname{rk} \underline{b} \otimes \underline{c} = 7.$$

(2). The first part of this example illustrates a common failure of the upper semi-continuity property of the ranks of partitions with respect to tensor product. Since this fails for partitions, it is reasonable to expect the same property to fail for maximal ranks of modules. The following is an explicit realization by kG-modules of this failure of upper semi-continuity. This example also shows that $M \otimes N$ can fail to have maximal rank at a π-point at which both M and N have maximal rank. This should be contrasted with the situation for maximal Jordan types ([8, 4.2]).

Let $G = \mathbb{G}_{a(1)}^{\times 2}$ so that $kG \simeq k[x,y]/(x^p, y^p)$. Consider the kG-module M of Example [8, 2.4], pictured as follows:

$$\begin{tikzcd}
& \bullet \arrow[dl, "y"'] \arrow[dr, "x"] & & \bullet \arrow[dl, "y"'] \arrow[dr, "x"] & & \bullet \arrow[dl, "y"'] \arrow[dr, "x"] & & \bullet \arrow[dl, "y"'] \arrow[dr, "x"] & \\
\bullet \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \\
& \bullet & & \bullet & & \bullet & & \bullet &
\end{tikzcd}$$

Recall that $\Pi(G) \simeq \operatorname{Proj} \mathrm{H}^\bullet(G,k) \simeq \mathbb{P}^1$. A point $[\lambda_1 : \lambda_2]$ on $\mathbb{P}^1$ is represented by a π-point $\alpha : k[t]/t^p \to kG$ such that $\alpha(t) = \lambda_1 x + \lambda_2 y$.

For $p > 5$, the module M has two Jordan types: the generic type $4[3] + 1[1]$ and the singular type $3[3] + 2[2]$, which occurs at $[1 : 0]$ and $[0 : 1]$ (see [8, 2.4]). Hence, M has constant rank. We compute possible local Jordan types of $M \otimes M$ using the fact that $\mu_{v*} : k(v)[t]/(t^p) \to k(v)G$ is a map of Hopf algebras for any $v \in V(G)$:

(i) $(4[3] + 1[1])^{\otimes 2} = 16[5] + 24[3] + 17[1]$,

(ii) $(3[3] + 2[2])^{\otimes 2} = 9[5] + 16[4] + 13[3] + 12[2] + 9[1]$.

By [18, 4.4], the first type is the generic Jordan type of $M \otimes M$. Hence, the generic (and maximal) rank of $M \otimes M$ is 112. On the other hand, the rank of the second type is 110. Hence, the rank of M at the points $[1 : 0], [0 : 1]$ is maximal, but the rank of $M \otimes M$ is not.

(3). Yet another result in [8], a direct consequence of the result on the tensor products of maximal types mentioned in (2), states that a tensor product of modules of constant Jordan type is a module of constant Jordan type. This distinguishes the family of modules of constant Jordan type from the modules of constant rank, for which this property fails. Let M be the same as in (2). The calculation above shows that M is of constant rank but $M \otimes M$ is not.

We also give an example of a different nature, avoiding point by point calculations of Jordan types. This example was pointed out to us by the referee. Let M be a cyclic kG-module of dimension less than p (e.g., $M = k[x,y]/(x^2, y^p)$). We have a short exact sequence $0 \to \Omega M \to kG \to M \to 0$. This implies that the local Jordan type of ΩM necessarily has p blocks, and, hence, ΩM has constant rank. Since $\Omega M \otimes \Omega^{-1} k \simeq M \oplus [\text{proj}]$, we conclude that the tensor product of two modules of constant rank produces a module which is not of constant rank.

4. Refined support varieties for arbitrary finite group schemes

In this section, we introduce the non-maximal support varieties $\Gamma^j(G)_M$ for an arbitrary finite group scheme, finite dimensional kG-module M, and integer $j, 1 \leq j < p$. These are well defined thanks to Theorem 3.6. After verifying a few simple properties of these refined support varieties, we investigate various explicit examples.

Definition 4.1. Let G be a finite group scheme, and let M be a finite dimensional kG-module. Set

$$\Gamma^j(G)_M = \{[\alpha_K] \in \Pi(G) \mid \operatorname{rk}(\alpha_K(t^j), M_K) \text{ is not maximal}\},$$

the non-maximal j-rank variety of M.

Our first example demonstrates that $\{\Gamma^j(G)_M\}$ is a finer collection of geometric invariants than $\Pi(G)_M$.

Example 4.2. Let $G = \mathrm{GL}(3, \mathbb{F}_p)$ with $p > 3$. By [21] (see [18, 4.10]), the irreducible components of $\Pi(G)$ are indexed by the conjugacy classes of maximal elementary p-subgroups of G which are represented by subgroups of the unipotent group $U(3, \mathbb{F}_p)$ of strictly upper triangular matrices. There are 3 such conjugacy classes, represented by the following subgroups:

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} a, b \in \mathbb{F}_p \right\} \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} a, b \in \mathbb{F}_p \right\} \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} a, b \in \mathbb{F}_p \right\}$$

Let M be the second symmetric power of the standard 3-dimensional (rational) representation of G. Then the generic Jordan type of M indexed by the first of these maximal elementary abelian subgroups of G is $[3] + 3[1]$, whereas the Jordan types indexed by each of the other conjugacy classes of maximal elementary abelian p–subgroups are $[2] + 4[1]$.

Thus, $\Pi(G)_M = \Pi(G)$ provides no information about M.

On the other hand, $\Gamma(G)_M = \Gamma^1(G)_M = \Gamma^2(G)_M$ equals the union of the two irreducible components of $\Pi(G)$ corresponding to the second and third maximal elementary abelian p–subgroups, whereas $\Gamma^i(G)_M = \emptyset$ for $i > 2$.

Our second example shows that $\Gamma^i(G)_M$ and $\Gamma^j(M)$ can be different, proper subsets of $\Pi(G)$.

Example 4.3. In [18, 4.13] A. Suslin and the authors constructed an example of a finite group G and a finite dimensional G-module M, such that $\Pi(G) = X \cup Y$ has two irreducible components and the generic Jordan types of M at the generic points of X and Y respectively are incomparable. Let G and M satisfy this property, and let α_K and β_L be generic π-points of X and Y respectively. If $\mathrm{JType}(\alpha_K^*(M_K))$ and $\mathrm{JType}(\beta_L^*(M_L))$ are incomparable, then Lemma 1.4 implies that there exist $i \neq j$ such that $\mathrm{rk}(\alpha_K(t^i), M_K) > \mathrm{rk}(\beta_L(t^i), M_L)$ but $\mathrm{rk}(\alpha_K(t^j), M_K) < \mathrm{rk}(\beta_L(t^j), M_L)$. Hence, $\Gamma^i(G)_M$ is a proper subvariety that contains the irreducible component Y whereas $\Gamma^j(G)_M$ is a proper subvariety that contains the irreducible component X.

Our third example is a simple computation for a general finite group scheme. It provides another possible "pattern" for the varieties $\Gamma^i(G)_M$.

Example 4.4. Let $\zeta_1 \in \mathrm{H}^{n_1}(G, k)$ be an even dimensional class, and $\zeta_2 \in \mathrm{H}^{n_2}(G, k)$ be an odd dimensional class. Consider $L_{\underline{\zeta}} = L_{\zeta_1, \zeta_2}$, the kernel of the map

$$\zeta_1 + \zeta_2 : \Omega^{n_1} k \oplus \Omega^{n_2} k \to k$$

The local Jordan type of $L_{\underline{\zeta}}$ at a π-point α is given in the following table:

$$\begin{cases} r[p] + [p-1], & \alpha^*(\zeta_1) \neq 0 \\ r[p] + [p-2] + [1], & \alpha^*(\zeta_1) = 0,\ \alpha^*(\zeta_2) \neq 0 \\ (r-1)[p] + 2[p-1] + [1], & \alpha^*(\zeta_1) = \alpha^*(\zeta_2) = 0 \end{cases}$$

Hence, $\Gamma^1(G)_{L_{\underline{\zeta}}} = \ldots = \Gamma^{p-2}(G)_{L_{\underline{\zeta}}} = Z(\zeta_1)$, whereas $\Gamma^{p-1}(G)_{L_{\underline{\zeta}}} = Z(\zeta_1) \cap Z(\zeta_2)$, where $Z(\zeta_1)$ denotes the zero locus of a class $\zeta_1 \in \mathrm{H}^\bullet(G, k)$ and $Z(\zeta_2)$ for $\zeta_2 \in \mathrm{H}^{\mathrm{odd}}(G, k)$ is defined in (5.3).

We next verify a few elementary properties of $M \mapsto \Gamma^j(G)_M$. Some of them are analogous to the properties of $\Gamma(G)_M$ stated in Prop 1.8.

Proposition 4.5. *Let G be a finite group scheme and M a finite dimensional kG-module.*

1. *$\Gamma^j(G)_M$ is a proper closed subset of $\Pi(G)$ for $1 \leq j < p$.*
2. *$\Gamma^j(G)_M = \emptyset$ if and only if M has constant j-rank.*
3. *If M and N are stably isomorphic, then $\Gamma^j(G)_M = \Gamma^j(G)_N$*
4. *If M is a module of constant j-rank, then $\Gamma^j(G)_{M\oplus N} = \Gamma^j(G)_N$.*
5. *$\Gamma^j(G)_M = \Gamma^j(G)_{\Omega^2(M)}$.*
6. *$\Gamma(G)_M = \cup_{1\leq j<p}\Gamma^j(G)_M$.*
7. *If M has the Jordan type $m[p]$ at some generic π-point, then $\Gamma^1(G)_M = \ldots = \Gamma^{p-1}(G)_M = \Pi(G)_M$.*

Proof. By definition, $\Gamma^j(G)_M \subset \Pi(G)$ can never equal $\Pi(G)$, so it is a proper subvariety. Moreover, assertions (2) and (6) also immediately follow from definitions and Lemma 1.4. Assertion (4) follows from the additivity of ranks and of the functor $\alpha_K^* : KG-\mathrm{mod} \to K[t]/t^p-\mathrm{mod}$ induced by a π-point α_K. Property (3) is proved exactly as in the proof of Proposition 1.8(1).

For (5), observe that a π-point α_K induces an exact functor and hence commutes with the Heller operator Ω. The statement now follows from the observation that for $K[t]/t^p$-modules, applying Ω^2 does not change the stable Jordan type.

To prove that $\Gamma^j(G)_M \subset \Pi(G)$ is closed as asserted in (1), we repeat the proof of [18, 5.2] establishing that $\Gamma(G)_M$ is closed. Indeed, the reduction in that proof to the special case in which G is infinitesimal applies without change. The proof in the special case of G infinitesimal uses the affine scheme of 1-parameter subgroups; this proof applies with only one minor change: the set of equations on the ranks of powers of $f_A : A[t]/t^p \to \mathrm{End}_A(M)$ (in the notation of that proof) is replaced by the set of equations on rank of only one, the j-th, power of f_A.

If M is generically projective as in (7), then $\Gamma(G)_M = \Pi(G)_M$. Let $\alpha_K \notin \Gamma(G)_M$ so that the Jordan type of $\alpha_K^*(M)$ is $m[p]$, and let $\beta_L \in \Gamma(G)_M$. Let $\sum b_i[i]$ be the Jordan type of $\beta_L^*(M_L)$. The statement follows easily from the formula (1.4.1): we have

$$\mathrm{rk}(\alpha_K(t^j), M_K) = m(p-j) > \sum_{i=j+1}^{p} b_i(i-j) = \mathrm{rk}(\beta_L(t^j), M_L),$$

where the inequality in the middle follows by downward induction on j from the assumption $mp = \dim M = \sum\limits_{i=1}^{p} b_i i$. Thus, $\Gamma^j(G)_M = \Gamma(G)_M$ for each $j, 1 \leq j < p$. □

Example 4.6. We point out that the "natural" analog of 1.8(5) is not true for modules of constant rank. Namely, $\Gamma^1(G)_{M\otimes N}$ does not have to be equal to $\Gamma^1(G)_N$ for M of constant rank. Indeed, let M be as in Example 3.9. Then M has constant rank and $\Gamma^1(E)_M = \emptyset$. But $\Gamma^1(E)_{M\otimes M} \neq \emptyset$ since $M \otimes M$ is not a module of constant rank.

Using a recent result of R. Farnsteiner [12, 3.3.2], we verify below that the non-maximal subvarieties $\Gamma^i(G)_M \subset \Pi(G)$ of an indecomposable kG-module M do not change when we replace M by any N in the same component as M of the stable

Auslander-Reiten quiver of G. This is a refinement of a result of the J. Carlson and the authors [8, 8.7] which asserts that if M is an indecomposable module of constant Jordan type than any N in the same component of the stable Auslander-Reiten quiver of G as M is also of constant Jordan type.

Proposition 4.7. *Let k be an algebraically closed field, and G be a finite group scheme over k. Let $\Theta \subset \Gamma_s(G)$ be a component of the stable Auslander-Reiten quiver of G. For any two modules M, N in Θ, and any $j, 1 \leq j \leq p-1$,*

$$\Gamma^j(G)_M = \Gamma^j(G)_N$$

Proof. Recall that $\Pi(G)$ is connected. If $\dim \Pi(G) = 0$, then $\Pi(G)$ is a single point so that $\Gamma^j(G)_M$ is empty for any kG-module M.

Now, assume that $\Pi(G)$ is positive dimensional. Since k is assumed to be algebraically closed, to show that $\Gamma^j(G)_M = \Gamma^j(G)_N$, it's enough to show that their k-valued points are the same. For this reason, we shall only consider π-points defined over k.

Let M be a kG-module in the component Θ, and write the Jordan type of $\alpha^*(M)$ as $\sum_{i=1}^{p} \alpha_i(M)[i]$. By [12, 3.1.1], each component Θ determines non-negative integer valued functions d_i on the set of π-points (possibly different on equivalent π-points) and a positive, integer valued function f on the modules occurring in Θ such that

$$\begin{cases} \alpha_i(M) = d_i(\alpha) f(M) \text{ for } 1 \leq i \leq p-1 \\ \alpha_p(M) = \frac{1}{p}(\dim M - d_p(\alpha) f(M)) \end{cases} \tag{4.7.1}$$

Assume $[\beta] \in \Gamma^j(G)_M$, so that there exists a π-point $\alpha : k[t]/t^p \to kG$ such that $\operatorname{rk}\{\alpha^j(t), M\} > \operatorname{rk}\{\beta^j(t), M\}$. By (1.4.1), this is equivalent to

$$\sum_{j=i+1}^{p} \alpha_i(M)(i-j) > \sum_{j=i+1}^{p} \beta_i(M)(i-j).$$

Using formula (4.7.1), we rewrite this inequality as

$$\sum_{j=i+1}^{p-1} d_i(\alpha) f(M)(i-j) + \frac{1}{p}(\dim M - d_p(\alpha) f(M))(p-j) >$$

$$\sum_{j=i+1}^{p-1} d_i(\beta) f(M)(i-j) + \frac{1}{p}(\dim M - d_p(\beta) f(M))(p-j).$$

Simplifying, we obtain

$$\left(\sum_{j=i+1}^{p-1} d_i(\alpha)(i-j) - \frac{p-j}{p} d_p(\alpha)\right) f(M) > \left(\sum_{j=i+1}^{p-1} d_i(\beta)(i-j) - \frac{p-j}{p} d_p(\beta)\right) f(M). \tag{4.7.2}$$

Now, let N be any other indecomposable kG-module in the component Θ. Multiplying the inequality (4.7.2) by the positive, rational function $f(N)/f(M)$, we obtain the same inequality as (4.7.2) with M replaced by N. Thus, $[\beta] \in \Gamma^j(G)_N$. Interchanging the roles of M and N, we conclude that $\Gamma^j(G)_M = \Gamma^j(G)_N$. $\square$

For an infinitesimal group scheme G, the closed subvarieties $\Gamma^j(G)_M \subset \Pi(G)$ admit an affine version $V^j(G) \subset V(G)$ defined as follows

Definition 4.8. Let G be an infinitesimal group scheme, M a finite dimensional kG-module, and j a positive integer, $1 \leq j < p$. We define

$$V^j(G)_M = \{v \in V(G) |\ \mathrm{rk}(\theta_v^j, M_{k(v)}) \ \text{ is not maximal}\} \cup \{0\} \subset V(G).$$

(see §2 for notations). So defined, $V^j(G)_M - \{0\}$ equals $\mathrm{pr}^{-1}(\Gamma^j(G)_M)$, where $\mathrm{pr} : V(G) - \{0\} \to \Pi(G)$ is the natural (closed) projection (see [16]).

Remark 4.9. We can express $V^j(G)_M$ in terms of the locally closed subvarieties $V^{\underline{a}}(G)_M$ introduced in §2. Namely, $V^j(G)_M$ is the union of $V^{\underline{a}}(G)_M \subset V(G)$ indexed by the Jordan types $\underline{a}$ with $\sum_{i=1}^p a_i \cdot i = \dim(M)$ satisfying the condition that there exists some Jordan type $\underline{b}$ with $V^{\underline{b}}(G)_M \neq \{0\}$ and $\sum_{i>j}^p b_i(i-j) > \sum_{i>j}^p a_i(i-j)$.

Our first representative example of $V^j(G)_M$ is a continuation of (2.5).

Example 4.10. Let $G = \mathrm{GL}_{N(1)}$, let M be the standard representation of GL_N, and assume p does not divide N. Recall that $V(\mathrm{GL}_{N(1)}) \simeq \mathcal{N}_p$, where $\mathcal{N}_p$ is the p-restricted nullcone of the Lie algebra $\mathfrak{gl}_N$ ([24, §6]). The maximal Jordan type of M is $r[p] + [N - rp]$, where rp is the greatest non-negative multiple of p which is less or equal to N (see [18, 4.15]). The rank of the j^{th}-power of this matrix equals $r(p-j) + (N - rp - j)$ if $N - rp > j$ and $r(p-j)$ otherwise.

For simplicity, assume k is algebraically closed so that we need only consider k-rational points of $\mathcal{N}_p$. For any $X \in \mathcal{N}_p$, $\theta_X : M \to M$ is simply the endomorphism X itself. Consequently, if $N - rp \leq j$, $V^j(G)_M \subset \mathcal{N}_p$ consists of 0 together with those non-zero p-nilpotent $N \times N$ matrices with the property that their Jordan types have strictly fewer than r blocks of size p; if $N - rp > j$, then $V^j(G)_M$ consists of 0 together with $0 \neq X \in \mathcal{N}_p$ whose Jordan type is strictly less than $r[p] + [N - rp]$.

Hence, the pattern for varieties $V^j(M)$ in this case looks like

$$\emptyset \neq V^1(G)_M = \ldots = V^n(G)_M \subset V^{n+1}(G)_M = \ldots = V^{p-1}(G)_M \subset V(G)$$

where $n = N - rp$.

Computing examples of $V^j(G)_M$ is made easier by the presence of other structure. For example, if $G = \mathcal{G}_{(r)}$, the r^{th}-Frobenius kernel of the algebraic group $\mathcal{G}$ and if the kG-module M is the restriction of a rational $\mathcal{G}$-module, then we verify in the following proposition that $V^j(G)_M$ is $\mathcal{G}$-stable, and thus a union of $\mathcal{G}$-orbits inside $V(G)$.

Lemma 4.11. *Let $\mathcal{G}$ be an algebraic group, and let G be the r^{th} Frobenius kernel of $\mathcal{G}$ for some $r \geq 1$. If M is a finite dimensional rational $\mathcal{G}$-module, then each $V^j(G)_M$, $1 \leq j < p$, is a $\mathcal{G}$-stable closed subvariety of $V(G)$.*

Proof. Composition with the adjoint action of $\mathcal{G}$ on G determines an action

$$\mathcal{G} \times V(G) \ \to \ V(G).$$

Observe that for any field extension K/k and any $x \in \mathcal{G}(K)$, the pull-back of M_K via the conjugation action $\gamma_x : G_K \to G_K$ is isomorphic to M_K as a KG-module. Thus, the Jordan type of $(\mu \circ \epsilon)^*(M_K)$ equals that of $(\gamma_x \circ \mu \circ \epsilon)^*(M_K)$ for any 1-parameter subgroup $\mu : \mathbb{G}_{a(r),K} \to G_K$. □

Using Lemma 4.11, we carry out our second computation of $V^j(G)_M$ with G infinitesimal, this time for G of height 2.

Example 4.12. Let $G = \mathrm{SL}_{2(2)}$. For simplicity, assume k is algebraically closed. Recall that

$$V(G) = \{(\alpha_0, \alpha_1) \,|\, \alpha_1, \alpha_2 \in sl_2, \alpha_1^p = \alpha_2^p = [\alpha_1, \alpha_2] = 0\},$$

the variety of pairs of commuting p-nilpotent matrices ([23]). The algebraic group SL_2 acts on $V(G)$ by conjugation (on each entry).

Let $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. An easy calculation shows that the non-trivial orbits of $V(G)$ with respect to the conjugation action are parameterized by $\mathbb{P}^1$, where $[s_0 : s_1] \in \mathbb{P}^1$ corresponds to the orbit represented by the pair (s_0e, s_1e).

Let S_λ be a simple SL_2-module of highest weight λ, $0 \le \lambda \le p^2 - 1$. Since S_λ is a rational SL_2-module, the non-maximal rank varieties $V^j(G)_{S_\lambda}$ are SL_2-stable by Proposition 4.11. Hence, to compute the non-maximal rank varieties for S_λ it suffices to compute the Jordan type of S_λ at the orbit representatives (s_0e, s_1e). By the explicit formula ([17, 2.6.5]), the Jordan type of S_λ at (s_0e, s_1e) is given by the Jordan type of the nilpotent operator $s_1e + s_0^pe^{(p)}$ (here, $e^{(p)}$ is the divided power generator of $k\,\mathrm{SL}_{2(2)}$ as described in [17, 1.4]).

The non-maximal rank varieties $V^j(G)_{S_\lambda}$ depend upon which of the following three conditions λ satisfies.

(1) $\boxed{0 \le \lambda \le p - 1}$. In this case, the Jordan type of $e \in k\,\mathrm{SL}_{2(2)}$ as an operator on S_λ is $[\lambda + 1]$. On the other hand, the action of $e^{(p)}$ is trivial. Hence, if $j \ge \lambda + 1$, then the action $(s_1e + s_0^pe^{(p)})^j$ is trivial for any pair (s_0, s_1). For $1 \le j \le \lambda$, the j-rank is maximal (and equals $\lambda + 1 - j$) whenever $s_1 \ne 0$. We conclude that for $j > \lambda$, we have $V^j(G)_{S_\lambda} = 0$, and for $1 \le j \le \lambda$, $V^j(G)_{S_\lambda}$ is the orbit of $V(G)$ parametrized by $[1 : 0]$.

(2) $\boxed{p \le \lambda < p^2 - 1}$. Let $\lambda = \lambda_0 + p\lambda_1$. By the Steinberg tensor product theorem, we have $S_\lambda = S_{\lambda_0} \otimes S_{\lambda_1}^{(1)}$. Observe that e acts trivially on $S_{\lambda_1}^{(1)}$ and $e^{(p)}$ acts trivially on S_{λ_0}. Moreover, the Jordan type of $e^{(p)}$ as an operator on $S_{\lambda_1}^{(1)}$ is the same as the Jordan type of e as an operator on S_{λ_1}. Hence, the Jordan type of $s_1e + s_0^pe^{(p)}$ as an operator on $S_{\lambda_0} \otimes S_{\lambda_1}^{(1)}$ is $[\lambda_0 + 1] \otimes [\lambda_1 + 1]$ when $s_0s_1 \ne 0$. If $s_0 = 0$ or $s_1 = 0$ we get the types $[\lambda_0 + 1] \otimes (\mathrm{triv})$ or $(\mathrm{triv}) \otimes [\lambda_1 + 1]$ respectively.

   (a) For $0 < \lambda_0, \lambda_1 < p - 1$, the tensor product formula for Jordan types (see [8, Appendix]) implies that the j-rank of $[\lambda_0 + 1] \otimes [\lambda_1 + 1]$ is strictly greater than that of $[\lambda_0 + 1] \otimes (\mathrm{triv})$ or $(\mathrm{triv}) \otimes [\lambda_1 + 1]$ for $j \le \lambda_1 + \lambda_0$. Hence, the non-maximal j-rank variety in the case when $j \le \lambda_1 + \lambda_0$ is a union of two orbits, parameterized by $[1 : 0]$ and $[0 : 1]$. If $j > \lambda_1 + \lambda_0$, then the non-maximal j-rank variety is trivial since the j-rank is 0 at every point.

   (b) If $\lambda_0 = 0$, then $S_\lambda \simeq S_{\lambda_1}^{(1)}$. Hence, the computation for S_λ for $\lambda < p$ implies that the non-maximal j-rank variety in this case is the orbit corresponding to $[0 : 1]$ for $j \le \lambda_1$ and is trivial otherwise.

   (c) For $\lambda_0 = p - 1$ or $\lambda_1 = p - 1$, the non-maximal j-rank variety is the same as the support variety for any j, since the support variety is a

proper subvariety of $V(G)$ in this case. The support varieties for these modules were computed in [24, §7] (see also [17, 1.17(4)]).

(3) $\boxed{\lambda = p^2-1}$. In this case, S_λ is the Steinberg module for $\mathrm{SL}_{2(2)}$. Hence, it is projective, so the non-maximal rank varieties are all trivial.

We summarize our calculations in the table below. Let $\lambda = \lambda_0 + p\lambda_1$, and $\overline{\lambda} = \lambda_0 + \lambda_1$. If $j > \overline{\lambda}$, then $V^j(G)_{S_\lambda} = \emptyset$. For $j \le \overline{\lambda}$, we have

$$V^j(G)_{S_\lambda} = \begin{cases} \{(\alpha_0, 0)\} \cup \{(0, \alpha_1)\} & \text{if } 0 < \lambda_0, \lambda_1 < p-1 \\ \{(\alpha_0, 0)\} & \text{if } \lambda_0 \ne 0,\ \lambda_1 = 0 \text{ or } \lambda_0 = p-1, \lambda_1 \ne p-1 \\ \{(0, \alpha_1)\} & \text{if } \lambda_0 = 0,\ \lambda_1 \ne 0 \text{ or } \lambda_0 \ne p-1, \lambda_1 = p-1 \\ 0 & \text{if } \lambda_0 = \lambda_1 = p-1. \end{cases}$$

In particular, for a given $\lambda = \lambda_0 + p\lambda_1$ we get the following pattern for $M = S_\lambda$:

$$V(G) \supset V^1(G)_M = \cdots = V^{\bar{\lambda}}(G)_M \supset V^{\bar{\lambda}+1}(G)_M = \cdots = V^{p-1}(G)_M = \{0\}.$$

Observe that the only simple modules of constant rank are the trivial module and the Steinberg module. An interested reader may find it instructive to compare this calculation to the calculation of support varieties for $\mathrm{SL}_{2(2)}$ ([17, 1.17(4)], see also [24, §7]).

5. Subvarieties of $\Pi(G)$ associated to individual Ext-classes

For M a kG-module of constant rank, we associate to a cohomology class ζ in $\mathrm{H}^1(G, M)$ a closed subvariety $Z(\zeta) \subset \Pi(G)$ which generalizes the construction of the zero locus $Z(\zeta) \subset \operatorname{Spec} \mathrm{H}^\bullet(G, k)$ of a homogeneous cohomology class. We show that this construction is intrinsically connected to the non-maximal rank variety, and establish some "realization" results for non-maximal varieties as an application. Unless otherwise indicated, throughout this section G will denote an arbitrary finite group scheme over k.

Lemma 5.1. *Let M be a finite dimensional kG-module, and let ζ be a cohomology class in $\mathrm{H}^1(G, M)$. Consider the corresponding extension*

$$\tilde{\zeta} : 0 \to M \to E_\zeta \to k \to 0.$$

For any π-point $\alpha_K : K[t]/t^p \to KG$, the following are equivalent:

(i) *the cohomology class $\alpha_K^*(\zeta_K) \in \mathrm{H}^1(K[t]/t^p, M_K)$ is trivial.*

(ii) $\mathrm{rk}(\alpha_K^*(t), E_\zeta) \;=\; \mathrm{rk}(\alpha_K^*(t), M)$.

(iii) $\mathrm{JType}(\alpha_K^*(E_{\zeta,K})) \;=\; \mathrm{JType}(\alpha_K^*(M_K)) + 1[1]$.

Proof. Recall that $\alpha_K^*(-)$ is exact (by definition, α_K is flat); moreover, the sequence $\alpha_K^*(\tilde{\zeta})$ splits if and only if $\alpha_K^*(\zeta) = 0$ in $\mathrm{H}^1(K[t]/t^p, K)$. Thus, it suffices to prove that a short exact sequence $0 \to M \to E \to K \to 0$ of $K[t]/t^p$-modules splits if and only if $\mathrm{rk}(t, M) = \mathrm{rk}(t, E)$ if and only if $\mathrm{JType}(E) = \mathrm{JType}(M) + 1[1]$. Let $\underline{b} = \sum_{i=1}^p b_i[i]$ be the Jordan type of E and $\underline{a} = \sum_{i=1}^p a_i[i]$ be the Jordan type of M. Then this short exact sequence splits if and only if the map $E \to k$ factors through the summand $b_1[1]$ of E which occurs if and only if $b_i = a_i, i > 1$ which is equivalent to $\mathrm{rk}(t, M) = \mathrm{rk}(t, E)$. □

Proposition 5.2. *Let M be a kG-module of constant rank, and let ζ be a cohomology class in $\mathrm{H}^1(G, M)$. Consider the corresponding extension*

$$\tilde{\zeta} : 0 \to M \to E_\zeta \to k \to 0.$$

(1) *If E_ζ has constant rank equal to that of M, then $\alpha_K^*(\zeta_K) \in \mathrm{H}^1(K[t]/t^p, M)$ is trivial for every π-point $\alpha_K : K[t]/t^p \to KG$.*
(2) *If E_ζ has constant rank greater than that of M, then $\alpha_K^*(\zeta_K) \in \mathrm{H}^1(K[t]/t^p, M)$ is non-trivial for every π-point $\alpha_K : K[t]/t^p \to KG$.*
(3) *If E_ζ does not have constant rank, then $\alpha_K^*(\zeta)$ is trivial if and only if $[\alpha_K] \in \Gamma^1(G)_{E_\zeta} \subset \Pi(G)$.*
(4) *For any two equivalent π-points α_K, β_L of G, $\alpha_K^*(\zeta_K)$ is trivial if and only if $\beta_L^*(\zeta_L)$ is trivial.*

Proof. Assertions (1) and (2) follow immediately from Lemma 5.1. Assertion (3) also follows from Lemma 5.1: if E_ζ does not have constant rank, then the complement of $\Gamma^1(G)_{E_\zeta}$ in $\Pi(G)$ consists of those equivalence classes of π-points α_K satisfying Lemma 5.1(ii.).

To prove that the vanishing of $\alpha_K^*(\zeta_K)$ depends only upon the equivalence class of α_K, we examine each of the three cases considered above. In case (1), $\alpha_K^*(\zeta_K) = 0$ for all π-points α_K: on the other hand, in case (2) $\alpha_K^*(\zeta_K) \neq 0$ for all π-points α_K. Finally, the assertion in case (3) follows immediately from Theorem 3.6. ☐

Proposition 5.2(4) justifies the following definition.

Definition 5.3. For M a module of constant rank, and $\zeta \in \mathrm{H}^1(G, M)$, we define

$$Z(\zeta) \;\equiv\; \{[\alpha_K] \mid \alpha_K^*(\zeta) = 0\} \;\subset\; \Pi(G). \tag{5.3.1}$$

For $\zeta \in \mathrm{H}^m(G, k)$, we define

$$Z(\zeta) \;\equiv\; \{[\alpha_K] \mid \alpha_K^*(\zeta) = 0\} \;\subset\; \Pi(G). \tag{5.3.2}$$

Since $\mathrm{H}^m(G, k) \simeq \mathrm{H}^1(G, \Omega^{1-m}k)$, the definition of (5.3.2) is a special case of that of (5.3.1). For $m = 2n$ even, $Z(\zeta)$ corresponds under the isomorphism $\Pi(G) \simeq \operatorname{Proj} \mathrm{H}^\bullet(G, k)$ with the hypersurface $\langle \zeta = 0\rangle$ in $\operatorname{Spec} \mathrm{H}^\bullet(G, k)$ as shown below in Proposition 5.7.

Remark 5.4. We point out that Definition 5.3 is not as straight-forward as it might appear.

- Let $G = \mathbb{Z}/p \times \mathbb{Z}/p$ with $p > 2$, write $kG = k[x, y]/(x^p, y^p)$ and consider $M = kG/(x - y^2)$ as in Example 2.11. Consider the short exact sequence
$$0 \to \operatorname{Rad}(M) \to M \to k \to 0,$$
with associated extension class $\zeta \in \mathrm{H}^1(G, \operatorname{Rad}(M))$. Consider the equivalent π-points $\alpha, \alpha' : k[t]/t^p \to kG$ of Example 2.11. Then, $\alpha^*(\zeta) \neq 0$, yet $\alpha'^*(\zeta) = 0$. Thus, the "zero locus" of ζ is not a well defined subset of $\Pi(G)$.
- Let $\zeta \in \mathrm{H}^{2n}(G, k)$ represented by $\hat{\zeta} : \Omega^{2n}k \to k$. By definition of L_ζ, we have an extension
$$\tilde{\xi} :\; 0 \to L_\zeta \to \Omega^{2n}k \xrightarrow{\hat{\zeta}} k \to 0,$$
corresponding to a cohomology class $\xi \in \mathrm{H}^1(G, L_\zeta)$. Then for any π-point $\alpha_K : K[t]/t^p \to KG$, $\alpha_K^*(\tilde{\xi})$ splits if and only if $\alpha_K^*(L_\zeta)$ is free if and only if $[\alpha_K] \notin \Pi(G)_{L_\zeta}$ if and only of $\alpha_K^*(\zeta) \neq 0$. Thus, the zero locus of ξ equals the *complement* of the zero locus of ζ (and thus is open in $\Pi(G)$).

- For $\zeta \in \mathrm{H}^{2n+1}(G,k)$, one could define $Z(\zeta)$ as the zero locus of the Bockstein of ζ provided one is in a situation in which the Bockstein is defined and well behaved. See the discussion of the Bockstein following Example 5.6.

We recall from [7] that a short exact sequence of kG modules

$$\tilde{\xi}: \quad 0 \to M \to E \to Q \to 0$$

is said to be *locally split* if $\alpha_K^*(\tilde{\xi})$ splits for every π-point $\alpha_K : K[t]/t^p \to KG$ of G.

Proposition 5.5. *Let M be a module of constant rank, and let ζ be a cohomology class in $\mathrm{H}^1(G,M)$. Consider the corresponding extension*

$$\tilde{\zeta} : 0 \to M \to E_\zeta \to k \to 0.$$

Then

$$Z(\zeta) = \begin{cases} \Pi(G), & \text{if } \tilde{\zeta} \text{ is locally split} \\ \Gamma^1(G)_{E_\zeta}, & \text{if } \tilde{\zeta} \text{ is not locally split.} \end{cases}$$

In particular, $Z(\zeta) \subset \Pi(G)$ is closed.

Proof. Observe that $\tilde{\zeta}$ is split at $[\alpha_K]$ if and only if $\alpha_K^*(\zeta) = 0$. We first consider ζ such that E_ζ has constant rank. Then by Proposition 5.2.1, $Z(\zeta)$ equals $\Pi(G)$ if $\tilde{\zeta}$ is locally split and $Z(\zeta) = \emptyset$ by Proposition 5.2.2 if $\tilde{\zeta}$ is not locally split. Alternatively, if E_ζ does not have constant rank, then Proposition 5.2.3 gives the asserted description of $Z(\zeta)$.

Because $\Gamma^1(G)_{E_\zeta} \subset \Pi(G)$ is closed by Proposition 4.5 and of course $\Pi(G)$ is itself closed in $\Pi(G)$, we conclude that $Z(\zeta)$ is closed inside $\Pi(G)$. $\square$

We remark that $\zeta \in \mathrm{H}^1(G,M)$ can be non-zero and yet $Z(\zeta) = \emptyset$. To say $Z(\zeta) = \emptyset$ is to say that $\alpha_K^*(\zeta) = 0$ for all π-points α_K. Consider, for example, an even dimensional non-trivial cohomology class $\zeta \in \mathrm{H}^{2n}(G,k)$ which is a product of odd dimensional classes. Since the product of any two odd classes in $\mathrm{H}^*(k[t]/t^p,k)$ is zero, $\alpha_K^*(\zeta) = 0$ for all π-points α_K of G. On the other hand, ζ can be identified with a cohomology class in $\mathrm{H}^1(G,\Omega^{1-2n}(k)) \simeq \mathrm{H}^{2n}(G,k)$. Since $\Omega^{1-2n}(k)$ is a module of constant Jordan type (see [8]), the class ζ satisfies the requirements of Proposition 4.5.

A more interesting example is the following.

Example 5.6. Let G be a finite group scheme with the property that the dimension of $\Pi(G)$ is at least 1. Let $\zeta' \in \widehat{\mathrm{H}}^{-i}(G,k)$, $i > 0$, be an element in the negative Tate cohomology of G. As shown in [8, 6.3], $\alpha_K^*(\zeta') = 0$ for any π-point α_K. Then ζ' corresponds to $\zeta \in \mathrm{H}^1(G,\Omega^{i+1}(k))$ under the isomorphism $\mathrm{H}^{-i}(G,k) \simeq \mathrm{H}^1(G,\Omega^{i+1}(k))$; by the naturality of this isomorphism, $\alpha_K^*(\zeta) = 0 \in \widehat{\mathrm{H}}^{-i}(K[t]/t^p,K)$ for any π-point α_K.

Thus, $\zeta \neq 0$, $\tilde{\zeta}$ is locally split, and $Z(\zeta) = \emptyset$ for this choice of $\zeta \in \mathrm{H}^1(G,\Omega^{i+1}(k))$.

For any field extension K/k, let $R_K = W_2(K)$ denote the Witt vectors of length 2 for K. Assume that G over k embeds into an R_k-group scheme G_{R_k} so that $G = G_{R_k} \times_{\operatorname{Spec} R_k} \operatorname{Spec} k \subset G_{R_k}$, thereby inducing by base change $G_K \subset G_{R_K}$. Then we may define the Bockstein $\beta : \mathrm{H}^i(G_K,K) \to \mathrm{H}^{i+1}(G_K,K)$ for $i > 0$ as the connecting homomorphism for the short exact sequence of G_{R_K}-modules

$$0 \to K \ \to \ R_K \ \to \ K \to 0. \tag{5.6.1}$$

(The reader is referred to [11, 3.4] for a discussion of this Bockstein.) Since any π-point $\alpha_K : K[t]/t^p \to KG$ lifts to a map $\tilde{\alpha}_K : R_K[t]/t^p \to R_K G_{R_K}$ of R-algebras, $\alpha^* : \mathrm{H}^*(G,K) \to \mathrm{H}^*(K[t]/t^p, K)$ commutes with this Bockstein. Since $\beta : \mathrm{H}^{2d-1}(K[t]/t^p, K) \to \mathrm{H}^{2d}(K[t]/t^p, K)$ is an isomorphism, we conclude that if $x \in \mathrm{H}^{2d-1}(G,k)$, then $\alpha_K^*(x)$ vanishes if and only if $\alpha_K^*(\beta(x)) = 0$, where $\beta(x) \in \mathrm{H}^{2d}(G,k)$. Thus, for such G lifting to G_{R_k} and for $p>2$, when considering $Z(\zeta)$ for homogeneous classes in $\mathrm{H}^*(G,k)$, it suffices to restrict attention to the subalgebra $\mathrm{H}^\bullet(G,k)$ of even dimensional classes.

As we see in the following proposition, Definition 5.3 of $Z(\zeta)$ extends the "classical" definition of the vanishing locus of a (homogeneous) cohomology class in $\mathrm{H}^\bullet(G,k)$.

Proposition 5.7. *Let n be a positive integer, and set $M = \Omega^{1-2n}(k)$. Let $\zeta \in \mathrm{H}^1(G,M)$, and let $\zeta' \in \mathrm{H}^{2n}(G,k)$ be the corresponding element under the natural isomorphism $\mathrm{H}^1(G,M) \simeq \mathrm{H}^{2n}(G,k)$. Then the isomorphism $\Pi(G) \simeq \operatorname{Proj} \mathrm{H}^\bullet(G,k)$ of Theorem 1.2 restricts to an isomorphism*

$$Z(\zeta) \;=\; \operatorname{Proj} \mathrm{H}^\bullet(G,k)/(\zeta').$$

Proof. Let $L_{\zeta'}$ be the Carlson module associated to the class ζ'. The exact triangle

$$\tilde{\zeta} : \Omega^{1-2n}(k) \to E_\zeta \to k \to \Omega^{-2n}(k)$$

corresponds to the exact triangle

$$\tilde{\zeta}' : \Omega^1(k) \longrightarrow L_{\zeta'} \longrightarrow \Omega^{2n}(k) \xrightarrow{\zeta'} k$$

under the shift Ω^{2n}. Hence, $L_{\zeta'}$ is stably isomorphic to $\Omega^{2n}(E_\zeta)$. By Prop. 4.5,

$$\Gamma^1(G)_{E_\zeta} = \Gamma^1(G)_{L_{\zeta'}}. \tag{5.7.1}$$

If $\tilde{\zeta}$ is locally split, then so is $\tilde{\zeta}'$ by the naturality of the isomorphism $\mathrm{H}^1(G,M) \simeq \mathrm{H}^{2n}(G,k)$. This implies that ζ' is nilpotent by the "Nilpotence detection theorem" of Suslin ([22]). Hence, in this case $\operatorname{Proj} \mathrm{H}^\bullet(G,k)/(\zeta') = \operatorname{Proj} \mathrm{H}^\bullet(G,k) \simeq \Pi(G)$. By Prop. 5.5, $Z(\zeta) \simeq \Pi(G)$ as well. Hence, in this case $Z(\zeta) = \Pi(G) \simeq \operatorname{Proj} \mathrm{H}^\bullet(G,k)/(\zeta')$.

If $\tilde{\zeta}$ is not locally split, then $Z(\zeta) = \Gamma^1(G)_{E_\zeta}$ by Proposition 5.5. Since $L_{\zeta'}$ is generically projective, Proposition 4.5 implies that $\Gamma^1(G)_{L_{\zeta'}} = \Pi(G)_{L_{\zeta'}}$. By [16, 2.9] (see [6] for finite groups), $\Pi(G)_{L'_\zeta} \simeq \operatorname{Proj} \mathrm{H}^\bullet(G,k)/(\zeta')$ under the isomorphism Φ_G of (1.2). The equality (5.7.1) now implies $Z(\zeta) \simeq \operatorname{Proj} \mathrm{H}^\bullet(G,k)/(\zeta')$. □

Proposition 5.8. *Let G be a finite group scheme over k. Let $\zeta_i \in \mathrm{H}^{2d_i+1}(G,k) \simeq \mathrm{H}^1(G, \Omega^{-2d_i}k)$, $1 \le i \le r, d_i \ge 0$. Let $M = \oplus_{i=1}^r \Omega^{-2d_i}k$, and set $\zeta = \oplus_i \zeta_i \in \mathrm{H}^1(G,M) = \oplus_i \mathrm{H}^1(G, \Omega^{-2d_i}k)$. Let*

$$0 \to M \to E_\zeta \to k \to 0$$

be the corresponding extension. Then

$$\Gamma^1(G)_{E_\zeta} \;=\; Z(\zeta) \;=\; Z(\zeta_1) \cap \ldots \cap Z(\zeta_r),$$

and $\Pi(G)_{E_\zeta} = \Pi(G)$.

Proof. To prove (1), observe that Lemma 5.1(1) implies that $\Gamma^1(G)_{E_\zeta} = \{[\alpha_K] \mid \alpha_K^*(\zeta) = 0\}$. Since $\zeta = \oplus\zeta_i$, we further conclude $\{[\alpha_K] \mid \alpha_K^*(\zeta) = 0\} = \{[\alpha_K] \mid \alpha_K^*(\zeta_i) = 0 \text{ for all } i\} = \bigcap_i Z(\zeta_i)$.

To verify that $\Pi(G)_{E_\zeta} = \Pi(G)$, we observe that the the generic Jordan type of E_ζ is of the form $m[p] + [2] + (r-1)[1]$ at generic points $[\alpha_K] \in \Pi(G)$ such that $\alpha_K^*(\zeta) \neq 0$ and of the form $m[p] + (r+1)[1]$ otherwise. This follows immediately from the observation that $\Omega^{-2d_i}(k)$ has constant Jordan type of the form $m_i[p]+[1]$, and thus M has constant (and, in particular, generic) Jordan type $(\sum_i m_i)[p]+r[1]$. □

As we see below, the construction of $E_{\underline{\zeta}}$ in Proposition 5.8 above is in fact a generalized Carlson module $L_{\underline{\zeta}}$ (as defined in [8]) "in disguise". This phenomenon has already appeared in the proof of Proposition 5.7 for a single cohomology class ζ. Since this construction applies to homogeneous cohomology classes ζ_i which are either all in even degree or all in odd degree, and since Proposition 5.8 discusses classes of odd degree, we consider in Example 5.9 classes ζ_i in even degree.

Example 5.9. Let $\underline{\zeta} = (\zeta_1, \dots, \zeta_r)$, where $\zeta_i \in \mathrm{H}^{2d_i}(G,k) \simeq \underline{\mathrm{Hom}}(\Omega^{2d_i}(k), k)$, $1 \leq i \leq r, d_i \geq 0$. Let $L_{\underline{\zeta}}$ be the kernel of the map $\zeta = \sum \zeta_i : \bigoplus \Omega^{2d_i}(k) \to k$, so that we have an exact sequence:

$$0 \longrightarrow L_{\underline{\zeta}} \longrightarrow \bigoplus \Omega^{2d_i}(k) \xrightarrow{\zeta_1 + \cdots + \zeta_r} k \longrightarrow 0$$

This short exact sequence represents an exact triangle in $\operatorname{stmod} kG$. Shifting the triangle by Ω^{-1} we obtain a triangle

$$k \longrightarrow \Omega^{-1}(L_{\underline{\zeta}}) \longrightarrow \bigoplus \Omega^{2d_i-1}(k) \longrightarrow \Omega^{-1}(k)$$

Hence, $\underline{\zeta}$ corresponds to a short exact sequence

$$0 \longrightarrow k \longrightarrow F_{\underline{\zeta}} \longrightarrow \bigoplus \Omega^{2d_i-1}(k) \longrightarrow 0$$

with the middle term stably isomorphic to $\Omega^{-1}(L_{\underline{\zeta}})$. Taking the dual of this short exact sequence, we obtain the analogue with even dimensional cohomology classes of the short exact sequence which defines E_ζ as in Proposition 5.8 (but for odd classes):

$$0 \longrightarrow \bigoplus \Omega^{1-2d_i} k \longrightarrow E_\zeta \longrightarrow k \longrightarrow 0\ .$$

Hence, E_ζ is stably isomorphic to $\Omega^{-1}(L^{\#}_{\underline{\zeta}})$.

Our final result extends the construction of closed zero loci to extension classes $\xi \in \mathrm{Ext}^n_G(N, M)$ with both M, N of constant Jordan type. In other words, Proposition 5.10 introduces the (closed) support variety $Z(\xi)$ of such an extension class.

Proposition 5.10. *Let G be a finite group scheme and N, M finite dimensional kG-modules of constant Jordan type. Let $\xi \in \mathrm{Ext}^n_G(N,M) \simeq \mathrm{Ext}^1(\Omega^{n-1}(N), M)$ for some $n \neq 0$, and consider the corresponding extension*

$$\tilde{\xi}: \quad 0 \to M \to E_\xi \to \Omega^{n-1}(N) \to 0.$$

(1) If $\alpha_K,\ \beta_L$ are equivalent π-points of G, then $\alpha_K^*(\tilde{\xi})$ splits if and only if $\beta_L(\tilde{\xi})$ splits.

(2) If

$$Z(\xi) \;\equiv\; \{[\alpha_K] \mid \alpha_K^*(\tilde{\xi}) \textit{ splits}\} \;\subset\; \Pi(G),$$

then

$$Z(\xi) = \begin{cases} \Pi(G), & \textit{if } \tilde{\xi} \textit{ is locally split} \\ \Gamma^1(G)_{E_\xi}, & \textit{if } \tilde{\xi} \textit{ is not locally split.} \end{cases}$$

Proof. There is a natural isomorphism

$$\mathrm{Ext}^1_G(\Omega^{n-1}(N), M) \;\simeq\; \mathrm{H}^1(G, (\Omega^{n-1}(N))^{\#} \otimes M)$$

sending the extension class ξ to the cohomology class $\zeta \in \mathrm{H}^1(G, (\Omega^{n-1}(N))^{\#} \otimes M)$ (where $(\Omega^{n-1}(N))^{\#}$ is the linear dual of $\Omega^{n-1}(N)$). Hence, $\alpha_K^*(\tilde{\xi})$ splits if and only $\alpha_K^*(\tilde{\zeta})$ splits for any π-point α_K of G.

By [9, 5.2], $(\Omega^{n-1}(N))^{\#}$ has constant Jordan type. Thus, by [9, 4.3], $(\Omega^{n-1}(N))^{\#} \otimes M$ also has constant Jordan type. Consequently, the assertion of the Proposition for ξ follows from Proposition 4.5 for ζ.

□

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